Sensitivity Analysis of the LMI-based $\mathcal{H}_\infty$ Control Problem

A.S. Yonchev*, P.Hr. Petkov*, N.D. Christov⁠† and M.M. Konstantinov‡

*Department of Systems and Control, Technical University of Sofia
1000 Sofia, Bulgaria; Email: ajonchev@mail.bg, php@tu-sofia.bg

†Laboratory of Automatics, Computer Engineering and Signal Processing
Lille University of Science and Technology, 59655 Villeneuve d’Ascq
France; Email: Nicolai.Christov@univ-lille1.fr

‡Department of Mathematics, University of Architecture, Civil Engineering
and Geodesy, 1046 Sofia, Bulgaria; Email: mmk_fte@uacg.bg

Abstract—Local perturbation bounds are obtained for the continuous-time $\mathcal{H}_\infty$ control problem based on linear matrix inequalities (LMI). The sensitivity analysis of the perturbed LMI is done by introducing a suitable slightly perturbed right-hand part. This approach leads to tight, condition number based perturbation bounds for the LMI solutions to the $\mathcal{H}_\infty$ control problem.

I. INTRODUCTION

In the last decade a number of papers have been published on the sensitivity of the $\mathcal{H}_\infty$ control problem [2]. These papers, however, consider exclusively the case of the Riccati-based $\mathcal{H}_\infty$ control problem. In contrast, in this paper we study the sensitivity of the LMI-based $\mathcal{H}_\infty$ control problem. We propose a new approach to the perturbation analysis of this problem via introducing a suitable right hand part in the considered matrix inequalities. Using this new perturbation technique we obtain local perturbation bounds for the the continuous-time LMI-based $\mathcal{H}_\infty$ control problem in terms of condition numbers with respect to the perturbations in the data.

We use the following notations: $\mathbb{R}^{m \times n}$ the space of real $m \times n$ matrices; $\mathbb{R}^n = \mathbb{R}^{n \times 1}$; $I_n$ the unit $n \times n$ matrix; $e_n$ the unit $n \times 1$ vector; $M^T$ the transpose of $M$; $M^\dagger$ the pseudo inverse of $M$; $\|M\|_2 = \sigma_{\text{max}}(M)$ the spectral norm of $M$, where $\sigma_{\text{max}}(M)$ is the maximum singular value of $M$; $\|M\|_F = \sqrt{\text{tr}(M^T M)}$ the Frobenius norm of $M$; $\|M\|_\infty := \sup_{R \succeq 0} \|M(s)\|_2$; $\|\cdot\|$ is any of the above norms; $\text{vec}(M) \in \mathbb{R}^{mn}$ the column-wise vector representation of $M \in \mathbb{R}^{m \times n}$; $\Pi_{m,n} \in \mathbb{R}^{mn \times mn}$ the vec-permutation matrix, such that $\text{vec}(M^T) = \Pi_{m,n} \text{vec}(M)$; $M \otimes P$ the Kronecker product of the matrices $M$ and $P$; $\text{vec}(MXP) = (P^T \otimes M) \text{vec}(X)$ column-wise vector representation of the multiplication $MXP$. The notation “:=” stands for “equal by definition”.

The paper is organized as follows. In Section II we shortly present the problem setup and objective. Section III describes the performed linear sensitivity analysis of the LMI based $\mathcal{H}_\infty$ control problem. Section IV presents a numerical example before we conclude in Section V with some final remarks.
where the matrix $\bar{A}$ respect to perturbations in optimal value of which are slightly perturbed. Linear, condition number based bounds for is aimed at determining bounds for system. The sensitivity analysis of the $H$ denote by $\Delta$ present a sensitivity analysis of the optimal may affect the accuracy of the matrices $R$ and $S$ and hence the accuracy of controller matrices. It is not clear up to the moment how LMI sensitivity is connected to the sensitivity of the given $\mathcal{H}_\infty$ suboptimal problem.

In what follows, we assume that $\gamma_{opt}$ is determined and present a sensitivity analysis of the optimal $\mathcal{H}_\infty$ control problem based on the LMI (3)-(5).

Suppose that the matrices $A, \ldots, D$ and the quantity $\gamma$ in (3), (4) are subject to perturbations $\Delta A, \ldots, \Delta \gamma_{opt}$ and denote by $R^* + \Delta R, S^* + \Delta S$ the solution of the perturbed LMI system. The sensitivity analysis of the $\mathcal{H}_\infty$ control problem is aimed at determining bounds for $\Delta R$ and $\Delta S$ near the optimal value of $\gamma$, as functions of the perturbations in the data $A, \ldots, D$ and $\gamma_{opt}$. In the next section we shall derive linear, condition number based bounds for $\Delta R$ and $\Delta S$ with respect to perturbations in $A, B_2, C_2, D_1, D_2$ and $\gamma_{opt}$.

III. LINEAR SENSITIVITY ANALYSIS

The essence of our approach is to perform sensitivity analysis of LMI (3) and (4) in a similar way as for proper matrix equations after introducing suitable right hand sides which are slightly perturbed.

Consider first LMI (4). Its structure allows us to analyze only the perturbed inequality

$$(N_{21} + \Delta N_{21})^T \times \begin{bmatrix} (A + \Delta A)^T (S + \Delta S) + (S + \Delta S)(A + \Delta A) & 0 \\ B_1^T (S + \Delta S) & 0 \\ 0 & (S + \Delta S) B_1 \\ 0 & -\gamma I - \Delta \gamma I \end{bmatrix} \times (N_{21} + \Delta N_{21}) := \bar{P}^* + \Delta \bar{P}_1 < 0 \quad (6)$$

where the matrix $\bar{P}^*$ is obtained using the nominal LMI

$$(N_{21}^T A^T S + S B_1 : C_1^T \quad B_1^T S - \gamma I : D_{11}^T \quad \ldots \quad \ldots \quad C_1 \quad D_{11} : -\gamma I)^T = \bar{P}^* < 0 \quad (7)$$

and $\Delta \bar{P}_1$ is due to the data and closed-loop performance perturbations, the rounding errors and the sensitivity of the interior point method that is used to solve the LMs.

Within first order terms the perturbed relation (6) may be written as

$$N_{21}^T W N_{21} + N_{21}^T W \Delta N_{21} + \Delta N_{21}^T W N_{21} + \Delta N_{21}^T W \Delta N_{21} \quad (8)$$

where

$$W = \begin{bmatrix} A^T S^* + S^* A + A^T \Delta S + \Delta S A + A^T S^* + S^* A & 0 \\ B_1^T S^* + B_1 \Delta S & 0 \\ 0 & S^* B_1 + \Delta S B_1 \\ 0 & -\gamma_{opt} I - \Delta \gamma_{opt} I \end{bmatrix}.$$
\[ \text{vec}(\Omega_S) = \begin{bmatrix} (I \otimes S^*) + (S^* \otimes I)\Pi_{n^2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\epsilon_{p3} \end{bmatrix} \times \begin{bmatrix} \text{vec}(\Delta A) \\ \Delta_{\text{opt}} \end{bmatrix} \]  

\[ : = \begin{bmatrix} V_{t1} \\ V_{t2} \end{bmatrix} \begin{bmatrix} \text{vec}(\Delta A) \\ \Delta_{\text{opt}} \end{bmatrix} \]  

(14)

and

\[ \mathcal{N}_{S11} = (\tilde{\mathcal{N}}_{21} \otimes I)\Pi_{(n+1),n^2} + (I \otimes \tilde{\mathcal{N}}_{21}). \]

Thus we have

\[ V_S \Delta s + V_{ts1} \text{vec}(\Delta A) + V_{ts2} \Delta_{\text{opt}} \mathcal{N}_{S11} \text{vec}(\Delta N_{21}) = \text{vec}(\tilde{\mathcal{P}}_1) \]  

(15)

where

\[ V_S = (\tilde{\mathcal{N}}_{21} \otimes \tilde{\mathcal{N}}_{21}^T) V_{t1} \]

\[ V_{ts1} = (\tilde{\mathcal{N}}_{21} \otimes \tilde{\mathcal{N}}_{21}^T) V_{t2}. \]

It is well known [6] that the perturbation bound for the projector \( \mathcal{N}_{21} \) may be written as

\[ \|\Delta \mathcal{N}_{21}\|_2 \leq \|\mathcal{C}_2, D_{21}\|_2 \|\Delta C_2, D_{D21}\|_2. \]  

(16)

Using the fact that \( \|\text{vec}(M)\|_2 = M\|_F \), we finally obtain the relative perturbation bound for \( S^* \)

\[ \|\Delta S\|_F \leq \frac{1}{\|S^*\|_F} \left( V_{ab1} \left\| \Delta A \right\|_F \frac{\Delta \gamma_{\text{opt}}}{\gamma_{\text{opt}}} + V_{ab2} \left\| \Delta \gamma_{\text{opt}} \right\|_F \right) \]

\[ + \frac{1}{\|S^*\|_F} \left( V_{cd} \left\| \Delta C_2, D_{21} \right\|_F + V_1 \left\| \Delta \tilde{P}_1 \right\|_F \right) \]  

(17)

where

\[ V_{ab1} = \frac{\|V_{t1}\|_2 \|V_{ts1}\|_2 \|A\|_F}{\|S^*\|_F} \]

\[ V_{ab2} = \frac{\|V_{t1}\|_2 \|V_{ts2}\|_2 \|A\|_F}{\|S^*\|_F} \]

\[ V_{cd} = \frac{\|V_{t1}\|_2 \|V_{ts1}\|_2 \|C_2, D_2\|_F}{\|S^*\|_F}. \]

are the relative condition numbers of LMI (4) with respect to the perturbations in the data.

In a similar way we can obtain a relative perturbation bound for the solution \( R^* \) of the LMI (3). In this case we consider the perturbed inequality

\[ (N_{12} + \Delta N_{12})^T \]

\[ \times \begin{bmatrix} (A + \Delta A)(R^* + \Delta R) + (R^* + \Delta R)(A + \Delta A)^T \\ C_1(R^* + \Delta R) \end{bmatrix} \]

\[ + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} (R^* + \Delta R)C_1 \\ -\gamma_{\text{opt}}I - \Delta \gamma_{\text{opt}}I \end{bmatrix} \}

\[ \times (N_{12} + \Delta N_{12}) := \tilde{\mathcal{Q}}^* + \Delta \tilde{\mathcal{Q}}_1 < 0 \]  

(18)

where

\[ \mathcal{N}_{12}^T \begin{bmatrix} AR^* + R^* A^T & R^* C_1^T \\ C_1 R^* & -\gamma_{\text{opt}} I \end{bmatrix} \mathcal{N}_{12} := \tilde{\mathcal{Q}} < 0. \]  

Here, instead of \( \Delta s \) and \( \Omega_S \) we have

\[ \Delta_R = \begin{bmatrix} A \Delta R + \Delta R A^T & \Delta RC_1^T \\ C_1 \Delta R & 0 \end{bmatrix} \]

\[ \Omega_R = \begin{bmatrix} \Delta AR^* + R^* \Delta A^T & 0 \\ 0 & -\Delta \gamma_{\text{opt}} I \end{bmatrix} \]

and thus

\[ \begin{bmatrix} V_S \Delta s + V_{ts1} \text{vec}(\Delta A) + V_{ts2} \Delta_{\text{opt}} \mathcal{N}_{S11} \text{vec}(\Delta N_{21}) = \text{vec}(\tilde{\mathcal{P}}_1) \]  

(19)

\[ \text{vec}(\Delta_R) = \begin{bmatrix} (R^* \otimes I) + (I \otimes R^*)\Pi_{n^2} & 0 \\ 0 & 0 \\ 0 & -\epsilon_{p3} \end{bmatrix} \times \begin{bmatrix} \text{vec}(\Delta A) \\ \Delta_{\text{opt}} \end{bmatrix} \]

\[ \text{vec}(\Omega_R) = \begin{bmatrix} I \otimes A + A \otimes I \\ C_1 \otimes I \\ I \otimes C_1 \end{bmatrix} \]  

(20)

Denote

\[ \tilde{\mathcal{Q}}^* = N_{12}^T \mathcal{Q}^* N_{12}, \mathcal{Q}^* N_{12} = \tilde{\mathcal{N}}_{12}, N_{12}^T \mathcal{Q}^* = \tilde{\mathcal{N}}_{12} \]

\[ N_{R1} = (N_{12}^T \otimes I)\Pi_{(n+1),n^2} + (I \otimes \tilde{\mathcal{N}}_{12}^*) \]

\[ T_t = (N_{12}^T \otimes N_{12}^T) T_{t1} = (N_{12}^T \otimes N_{12}^T) T_{t1} \]

\[ T_{tr2} = (N_{12}^T \otimes N_{12}^T) T_{t2}. \]

Having in mind that

\[ \|\Delta N_{12}\|_F \leq \||B_2^T, D_{212}^T||_F \|\Delta B_2^T, D_{D21}^T||_F \| \]

we obtain the relative perturbation bound for \( R^* \)

\[ \|\Delta R\|_F \leq \frac{1}{\|R^*\|_F} \left( T_{ac1} \frac{\|\Delta A\|_F}{\|A\|_F} + T_{ac2} \frac{\|\gamma_{\text{opt}}\|_F}{\|A\|_F} \right) \]

\[ + \frac{1}{\|R^*\|_F} \left( T_{bd} \frac{\|\Delta B_2^T, D_{D21}^T\|_F}{\|B_2^T, D_{21}^T\|_F} + T_1 \frac{\|\Delta \tilde{Q}_1\|_F}{\|\tilde{Q}_1\|_F} \right) \]  

(20)

where

\[ T_{ac1} = \frac{T_{t1}^T \|2\| T_{tr1}^T \|2\| A\|_F}{\|R^*\|_F} \]

\[ T_{ac2} = \frac{T_{t1}^T \|2\| T_{tr2}^T \|2\| \gamma_{\text{opt}}}{\|R^*\|_F} \]

\[ T_{bd} = \frac{T_{t1}^T \|2\| N_{R1}^T \|2\| B_2^T, D_{D21}^T\|_F}{\|R^*\|_F} \]

are the relative condition numbers of LMI (3).
IV. NUMERICAL EXAMPLE

Consider the continuous-time system (1) with

\[
A = \begin{bmatrix}
-10/m & -1/c/m \\
-k/m & c/m
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 & 0 & 0 \\
-pm & -pc/m & -pk/m
\end{bmatrix}
\]

\[
B_2 = \begin{bmatrix}
0 & 0 \\
1/m & 1
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
-k/m & -c/m \\
0 & c
\end{bmatrix}
\]

\[
C_2 = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \quad D_{11} = \begin{bmatrix}
-pm & -pc/m & -pk/m \\
0 & 0 & 0
\end{bmatrix}
\]

\[
D_{12} = \begin{bmatrix}
1/m & 0 \\
0 & 0
\end{bmatrix}, \quad D_{21} = \begin{bmatrix}
0 & 0 & 0
\end{bmatrix}
\]

and \( m = 3, c = 1, k = 2, pm = 0.4, pc = 0.2, pk = 0.3 \). The perturbations in the data are chosen as

\[
\Delta A = A \times 10^{-i}, \quad \Delta B_1 = B_1 \times 10^{-i}, \quad \Delta B_2 = B_2 \times 10^{-i}
\]

\[
\Delta C_1 = C_1 \times 10^{-i}, \quad \Delta C_2 = C_2 \times 10^{-i}, \quad \Delta D_{11} = D_{11} \times 10^{-i}
\]

\[
\Delta D_{12} = D_{12} \times 10^{-i}, \quad \Delta \gamma_{opt} = 10^{-i} \times \gamma_{opt}
\]

for \( i = 8, 7, \ldots, 4 \).

The perturbed solutions \( R^* + \Delta R \) and \( S^* + \Delta S \) are computed using the LMI Control Toolbox of MATLAB [5]. The optimal closed-loop performance obtained is \( \gamma_{opt} = 0.4191 \). The relative perturbations in the solutions \( R^* \) and \( S^* \) of (3), (4) are estimated using the perturbation bounds (20) and (17), respectively.

The results obtained for different values of \( i \) are shown in the following table:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \frac{|\Delta R|<em>{\infty}}{|R^*|</em>{\infty}} )</th>
<th>Bound(17)</th>
<th>( \frac{|\Delta S|<em>{\infty}}{|S^*|</em>{\infty}} )</th>
<th>Bound(20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.2 \times 10^{-4}</td>
<td>3.8 \times 10^{-4}</td>
<td>1.0 \times 10^{-3}</td>
<td>1.5 \times 10^{-3}</td>
</tr>
<tr>
<td>7</td>
<td>1.7 \times 10^{-4}</td>
<td>3.8 \times 10^{-4}</td>
<td>1.9 \times 10^{-3}</td>
<td>1.5 \times 10^{-3}</td>
</tr>
<tr>
<td>6</td>
<td>4.1 \times 10^{-4}</td>
<td>3.8 \times 10^{-4}</td>
<td>8.2 \times 10^{-4}</td>
<td>1.5 \times 10^{-3}</td>
</tr>
<tr>
<td>5</td>
<td>1.9 \times 10^{-3}</td>
<td>3.8 \times 10^{-4}</td>
<td>9.9 \times 10^{-3}</td>
<td>1.5 \times 10^{-4}</td>
</tr>
<tr>
<td>4</td>
<td>2.0 \times 10^{-3}</td>
<td>3.8 \times 10^{-4}</td>
<td>1.0 \times 10^{-3}</td>
<td>1.5 \times 10^{-3}</td>
</tr>
</tbody>
</table>

V. CONCLUSIONS

Linear sensitivity analysis of the LMI arising in the continuous-time \( H_\infty \) control problem is done. Condition number based perturbation bounds are obtained in a similar way as for matrix equations, introducing a slightly perturbed right hand side in LMI. A numerical example is presented illustrating the accuracy of the proposed LMI perturbation bounds.

REFERENCES